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NON-TANGENTIAL SUMMABILITY OF POWER EXPANSIONS OF FUNCTIONS OF HARDY CLASSES

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Abstract: The class H_λ^p of λ-means of the power expansions of analytic functions $\varphi \in H^p$ is constructed. The behavior of λ-means for tending of arbitrary point (y, r) of the circle to a point $(x, 1)$ along non-tangential paths are studied. The estimates for the corresponding maximal operators and theorems of summability almost everywhere are established. The results for Hardy classes of functions are linked to performance $H_\lambda^p = L_\lambda^p \oplus i\tilde{L}_\lambda^p$ in which L_λ^p and $\tilde{L}_\lambda^p$, respectively, are classes of λ-means of Fourier series and conjugate series of $f = \operatorname{Re}\varphi$.

Introduction. Formulation of the problem

Denote $Q = [-\pi, \pi]$; let $\varphi = \varphi(z)$ be function of complex variable $z = r\exp(ix)$, analytic in a circle $\{z : |z| < 1\}$, $\operatorname{Im}\varphi(0) = 0$, $0 < r < 1, x \in Q$ and

$$\|\varphi\|_p = \sup_{0 \le r < 1} \int_Q |\varphi(r\exp(ix))|^p \, dx < \infty, \quad p \ge 1.$$

Class $H^p = H^p(Q)$, $H = H^1(Q)$ of such functions is called the Hardy class [1, vol. 1, p. 431]. In addition to H^p consider the Lebesgue class $L^p = L^p(Q)$ of 2π-periodic functions of a real variable, for which

$$\|f\|_p = \left(\int_Q |f(x)|^p \, dx \right)^{1/p} < \infty, \quad p \ge 1;$$

set $L = L(Q) = L^1(Q)$. The behavior of the power series

$$\varphi(r\exp(ix)) = \sum_{k=0}^{\infty} \mu_k(\varphi) r^k \exp(ikx) \qquad (1)$$

of functions $\varphi \in H^p$, $p \geq 1$ on the circle of convergence, when $r \to 1-0$, is well studied. So [2, p. 541]

$$\varphi(\exp(ix)) = \lim_{r \to 1-0} \varphi(r \exp(ix)) = f(x) + ig(x), \tag{2}$$

existsalmosteverywhere. Here $f, g \in L^p$ and the coefficients in the expansion (1) can be founded as

$$\mu_k(\varphi) = \frac{1}{2\pi} \int_Q \varphi(\exp(it)) \exp(-ikt)\, dt, \quad k = 0, 1, \ldots; \tag{3}$$

it is natural to assume that $\mu_k(\varphi) = 0$ for $k < 0$. If we put

$$\varphi(\exp(ix)) \sim \sum_{k=0}^{\infty} \mu_k(\varphi) \exp(ikx), \tag{4}$$

then (1) can be considered as a family of Abel-Poisson means of power series (4) on the circle of convergence. A significant strengthening of the result (2) is as follows [1, vol. 1, p. 438]. For every function $\varphi \in H^p$ the limiting values

$$\varphi(\exp(ix)) = \lim_{(y,r) \to (x,1)} \varphi(r \exp(ix))$$

exist for almost all x, if the point (y, r) is tending to $(x, 1)$, staying in the "corner" area, characterized by the condition

$$\frac{|y - x|}{1 - r} \leq d, \; d = \text{const}, \; d > 0$$

(tending along non-tangential paths). In the future, it will be convenient to change the designation of taking $h = \ln \frac{1}{r}$; so $h \sim 1 - r$ $(r \to 1 - 0)$. Non-tangential paths for $(y, h) \to (x, +0)$ will be now the paths within

$$\Gamma_d(x) = \left\{ (y, h) \mid y \in [-\pi, \pi], \; 0 < h < 1, \; \frac{|y - x|}{h} \leq d \right\}, \; d = \text{const}, \; d > 0 .$$

This approach to the behavior of the power expansions of function $\varphi \in H^p$ on the boundary of circle of convergence can be extended as follows. Let

$$\lambda = \{\lambda_k(h), \; k = 0, 1, \ldots; \; \lambda_0(h) = 1\} \tag{5}$$

be an arbitrary sequence infinite, generally speaking, determined by values of parameter $h > 0$. In this paper we study the behavior of λ-means

$$\Theta_h(\varphi) = \Theta(\varphi, y; \lambda, h) = \sum_{k=0}^{\infty} \mu_k(\varphi) \lambda_k(h) \exp(iky) \tag{6}$$

of series (4) for $(y, h) \to (x, 0)$, $(y, h) \in \Gamma_d(x)$. The main results will be linked to the so-called estimates of weak and strong type of maximal operators generated by λ-means of (4), λ-means of Fourier series and conjugate series of functions $f \in L^p(Q)$, $p \geq 1$, which are, respectively

$$U_h(f) = U(f, y; \lambda, h) = \sum_{k=-\infty}^{\infty} \lambda_{|k|}(h) c_k(f) \exp(iky) \tag{7}$$

and

$$\widetilde{U}_h(f) = \widetilde{U}(f, y; \lambda, h) = -i \sum_{k=-\infty}^{\infty} (\operatorname{sgn} k) \lambda_{|k|}(h) c_k(f) \exp(iky) . \qquad (8)$$

Here $\{c_k(f)\}$ is a sequence of complex Fourier coefficients

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \exp(-ikt)\, dt , \quad k = 0, \pm 1, \pm 2,... \qquad (9)$$

Denote H_λ^p , L_λ^p and $\widetilde{L}_\lambda^p$ the classes of means (6), (7) and (8), respectively. The results for Hardy classes of functions will be linked to performance

$$H_\lambda^p = L_\lambda^p \oplus i\, \widetilde{L}_\lambda^p .$$

Specifically [1, vol. 1, p. 173], the relation

$$\varphi(r \exp(iy)) = \sigma_r(f, y) + i \widetilde{\sigma}_r(f, y) \qquad (10)$$

holds for all $\varphi \in H^p$ and $\operatorname{Re} \varphi = f \in L^p$, $p \ge 1$ where $\sigma_r(f, y)$, $\widetilde{\sigma}_r(f, y)$ are the Poisson-Abel means and the conjugate means, corresponding to the case of $\lambda_k(h) = \exp(-hk)$ (still, $h = \ln \frac{1}{r}$). A more general (than (10)) statement

$$\Theta(\varphi, y; \lambda, h) = U(f, y; \lambda, h) + i \widetilde{U}(f, y; \lambda, h) \qquad (11)$$

was established in Theorem 3.1 of [3] by the using of (3) and arguments of type [2, p. 542-545].

Maximal operators

Introduce the following operators:

$$f \mapsto f^*, \text{ where } f^* = f^*(x) = \sup_{\eta > 0} \frac{1}{\eta} \int_{x-\eta}^{x+\eta} | f(t) |\, dt , \qquad (12)$$

(Hardy Littlewood maximal operator) and

$$f \mapsto \widetilde{f}\,^*, \text{ where } \widetilde{f}\,^* = \widetilde{f}\,^*(x) = \sup_{\eta > 0} \Big| \int_{\eta \le |t| \le \pi} \frac{f(x+t)}{2 \operatorname{tg} \frac{t}{2}} dt \Big| . \qquad (13)$$

Operators (12) and (13) are defined [1, vol. 1, p. 60, 401-402, 442, 443] for every $f \in L$; moreover, in this case there is almost everywhere a conjugate function

$$\widetilde{f}(x) = -\frac{1}{\pi} \lim_{\eta \to +0} \int_{\eta \le |t| \le \pi} f(x+t)\, ctg \frac{t}{2} dt .$$

For each $p \ge 1$ the following estimates of "weak type"

$$\mu\left\{x \in Q \mid f^*(x) > \varsigma > 0\right\} \le C_p \left(\frac{\| f \|_p}{\varsigma} \right)^p , \ \mu\{x \in Q \mid \widetilde{f}\,^*(x) > \varsigma > 0\} \le C_p \left(\frac{\| f \|_p}{\varsigma} \right)^p \qquad (14)$$

hold; here μ is the Lebesgue measure of corresponding sets. Along with (14) the estimates of "strong type" [1, vol. 1, p. 58-59, 404]

$$\| f^*(x) \|_p + \| \tilde{f}^*(x) \|_p \le C_p \| f \|_p, \quad p > 1$$

(the boundedness of operators (12) and (13) from L^p in L^p for all $p > 1$) and

$$\| f^*(x) \| + \| \tilde{f}^*(x) \| \le C(1 + \| f(\ln^+ | f |) \|) ;$$

$$\| f^*(x) \|_p + \| \tilde{f}^*(x) \|_p \le C_p \| f \|, \quad 0 < p < 1$$

are valid too. Here and below C will represent constants, which depend only on clearly specified indexes.

In accordance with λ-means (6) – (8), introduced above, we define the following maximal operators:

$$\varphi \mapsto \Theta_*(\varphi), \text{ where } \Theta_*(\varphi) = \Theta_*(\varphi, x; \lambda) = \sup_{(y,h)\in\Gamma_d(x)} | \Theta(\varphi, x; \lambda, h) | ; \tag{15}$$

$$f \mapsto U_*(f), \text{ where } U_*(f) = U_*(f, x; \lambda) = \sup_{(y,h)\in\Gamma_d(x)} | U(f, x; \lambda, h) | ; \tag{16}$$

$$f \mapsto \tilde{U}^*(f), \text{ where } \tilde{U}^*(f) = \tilde{U}^*(f, x; \lambda) = \sup_{(y,h)\in\Gamma_d(x)} | \tilde{U}(f, x; \lambda, h) | . \tag{17}$$

For each $h > 0$ denote $m = \left[\dfrac{1}{2dh}\right]$. The basis of the results of the behavior of means (6) – (8) is the following statement.

Theorem 1. Let the sequence (5) decreases so rapidly that

$$N | \lambda_N(h) | + N^2 | \Delta\lambda_N(h) | = o(1), \; N \to \infty, \tag{18}$$

and there is a constant $C = C_\lambda$ such that

$$\sum_{k=1}^{\infty} \frac{k(k+m)}{m} | \Delta^2 \lambda_k(h) | \le C. \tag{19}$$

Then, for all $f \in L(Q)$ the estimates

$$U^*(f, x; \lambda) \le C_\lambda f^*(x), \tag{20}$$

$$\tilde{U}^*(f, x; \lambda) \le C_\lambda \left(f^*(x) + \tilde{f}^*(x) \right) \tag{21}$$

hold.

Auxiliary assertion

Consider [1, vol. 1, p. 86, 153] the conjugate Dirichlet kernel

$$\tilde{D}_k(t) = \sum_{\nu=1}^{k} \sin \nu t = \frac{1}{2\operatorname{tg}\frac{1}{2}t} - \frac{\cos(k+\frac{1}{2})t}{2\sin\frac{1}{2}t} \tag{22}$$

and the conjugate Fejer kernel

$$\widetilde{F}_k(t)=\frac{1}{k+1}\sum_{\nu=0}^{k}\widetilde{D}(t)=\frac{1}{2\operatorname{tg}\frac{1}{2}t}-\ \widetilde{\widetilde{F}}_k(t),\ \text{where } \widetilde{\widetilde{F}}_k(t)=\frac{\cos(k+1)t}{2(k+1)\sin^2\frac{1}{2}t}; \qquad (23)$$

$k=0,1,...;\quad \widetilde{D}_0(t)=\widetilde{F}_{-1}(t)=0$.

Lemma. For all $k=0,1,...$ and $(y,h)\in\Gamma_d(x)$ the estimate

$$\left|\int_{-\pi}^{\pi} f(t)\ \widetilde{F}_k(y-t)\,dt\right| \le C\left(1+\frac{k}{m}\right)\left(f^*(x)+\widetilde{f}^*(x)\right) \qquad (24)$$

holds.

Proof. Let's start with a few comments. At $k=0$ the left side of (24) vanishes, so consider $k=1,2,...$

If $(y,h)\in\Gamma_d(x)$, then, obviously, $|y-t|\ge|x-t|-dh$. Hence, for x and t, such, that

$$|x-t|\ge\frac{1}{m}\ge 2dh \qquad (25)$$

the estimate

$$|y-t|\ge\frac{1}{2}|x-t| \qquad (26)$$

is valid. Indeed, (26) follows (cf. (25)) from inequality

$$|y-t|\ge|x-t|-dh\ge\frac{1}{2}|x-t|$$

for all $(y,h)\in\Gamma_d(x)$. Then, by definitions (23), the estimates

$$\left|\widetilde{F}_k(t)\right|\le Ck, \qquad |t|\le\pi; \qquad (27)$$

$$\left|\widetilde{\widetilde{F}}_k(t)\right|\le C\frac{1}{kt^2},\ 0<|t|\le\pi \qquad (28)$$

hold.

Assume firstly $k\le m$ and obtain the relation (24). By (27) and (28) we have

$$\left|\int_{-\pi}^{\pi} f(t)\widetilde{F}_k(y-t)\,dt\right| = \left|\int_{x-\pi}^{x+\pi} f(t)\widetilde{F}_k(y-t)\,dt\right| = \left|\int_{|x-t|\le\pi} f(t)\widetilde{F}_k(y-t)\,dt\right| \le$$

$$\le C\left(k\int_{|x-t|\le\frac{1}{k}}|f(t)|dt+\left|\int_{\frac{1}{k}\le|x-t|\le\pi} f(t)\operatorname{ctg}\frac{y-t}{2}dt\right|+\int_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\left|\widetilde{\widetilde{F}}_k(y-t)\right|dt=\right.$$

$$\left.=C(J_1(x,k)+J_2(x,k)+J_3(x,k)\right). \qquad (29)$$

It is obvious that

$$J_1(x,k)\le f^*(x). \qquad (30)$$

Further,

$$J_2(x,k)=\left|\int\limits_{\frac{1}{k}\le|x-t|\le\pi} f(t)\operatorname{ctg}\frac{x-t}{2}dt+\int\limits_{\frac{1}{k}\le|x-t|\le\pi} f(t)\frac{\sin\frac{x-y}{2}}{\sin\frac{x-t}{2}\sin\frac{y-t}{2}}dt\right|.$$

Taking into account (26), we have

$$|J_2(x,k)|\le C\left(\tilde{f}^*(x)+h\int\limits_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\frac{1}{(x-t)^2}dt\right).$$

Here [4]

$$\int\limits_{\frac{1}{k}\le|x-t|\le\pi}|f(t)|\frac{1}{(x-t)^2}dt=\int\limits_{\frac{1}{k}\le|t|\le\pi}|f(x+t)|\frac{1}{t^2}dt\le$$

$$\le Ck\sum_{j=1}^{S}\frac{k}{\left(2^{j-1}\right)^2}\int\limits_{\frac{2^{j-1}}{k}\le t\le\frac{2^j}{k}}|f(x+t)|\,dt\le C\;k\,f^*(x),\qquad(31)$$

if a positive integer S chosen from the condition

$$\frac{2^{S-1}}{k}\le\pi<\frac{2^S}{k}.$$

Hence

$$|J_2(x,k)|\le C\left(\tilde{f}^*(x)+\frac{k}{m}f^*(x)\right)\le C\left(\tilde{f}^*(x)+f^*(x)\right).\qquad(32)$$

Finally, in view of (26) and (31)

$$J_3(x,k)\le Cf^*(x).\qquad(33)$$

Now, according to (29), (30), (32), (33), the estimate (24) is valid at all $k\le m$.

Consider now the case of $k>m$. By (29) we have

$$\left|\int_{-\pi}^{\pi}f(t)\tilde{F}_k(y-t)\,dt\right|\le C\left(\int\limits_{|x-t|\le 1/m}|f(t)|k\,dt+\left|\int\limits_{\frac{1}{m}\le|x-t|\le\pi}f(t)\operatorname{ctg}\frac{y-t}{2}dt\right|+\right.$$

$$\left.+\int\limits_{\frac{1}{m}\le|x-t|\le\pi}|f(t)|\left|\tilde{\tilde{F}}_k(y-t)\right|dt\right)=C\left(\frac{k}{m}J_1(x,m)+J_2(x,m)+\mathrm{I}(x,k,m)\right).\qquad(34)$$

According to (30) and (32) we obtain

$$J_1(x,m)\le f^*(x),\qquad|J_2(x,m)|\le C\left(\tilde{f}^*(x)+f^*(x)\right).$$

Further, in view of (28) and (31)

$$\mathrm{I}(x,k,m))\le C\frac{m}{k}f^*(x)\le C\,f^*(x).$$

It follows now from (34) that

$$\left|\int_{-\pi}^{\pi} f(t)\widetilde{F}_k(y-t)\,dt\right| \le C(1+\frac{k}{m})\left(\widetilde{f}^*(x)+ f^*(x)\right)$$

for all $k > m$.

Thus, the estimate (24) is valid for all k = 1, 2, ..., and lemma is proved.

Proof of theorem 1

We shall prove (21), the proof of (20) is similar to (or even easier, [5]). Applying (9), Abel transform twice [1, vol. 1, p. 15], the obvious estimate $|D_N(t)| \le N$, N =1, 2,…, (cf. (22)), and (27), we obtain

$$\left|\widetilde{U}(f,y;\lambda,h)\right| = \left|\lim_{N\to+\infty}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left\{\sum_{k=1}^{N}\lambda_k(h)\sin k(y-t)\right\}dt\right| =$$

$$=\left|\frac{1}{\pi}\lim_{N\to+\infty}\left\{\lambda_N(h)\int_{-\pi}^{\pi} f(t)\ \widetilde{D}_N(y-t)\,dt + N\Delta\lambda_{N-1}(h)\int_{-\pi}^{\pi} f(t)\ \widetilde{F}_{N-1}(y-t)\,dt +\right.\right.$$

$$\left.\left.+\sum_{k=1}^{N-2}(k+1)\Delta^2\lambda_k(h)\int_{-\pi}^{\pi} f(t)\widetilde{F}_k(y-t)\,dt\right\}\right| \le C\lim_{N\to+\infty}\left\{\left(N\,|\lambda_N(h)| + N^2\,|\Delta\lambda_N(h)|\right);\right.$$

$$\left.\int_{-\pi}^{\pi}|f(t)|\ dt + \sum_{k=1}^{N-2}(k+1)\,|\Delta^2\lambda_k(h)|\left|\int_{-\pi}^{\pi} f(t)\widetilde{F}_k(y-t)\ dt\right|\right\}.$$

Due (18) it follows that

$$\left|\widetilde{U}(f,y;\lambda,h)\right| \le C\sum_{k=1}^{\infty}(k+1)\left|\Delta^2\lambda_k(h)\right|\left|\int_{-\pi}^{\pi} f(t)\widetilde{F}_k(y-t)\ dt\right|.$$

According to (24), we obtain

$$\left|\widetilde{U}(f,y;\lambda,h)\right| \le C\left(\widetilde{f}^*(x)+ f^*(x)\right)\sum_{k=1}^{\infty}\left|\Delta^2\lambda_k(h)\right|\ k\left(1+\frac{k}{m}\right),$$

and, because of the condition (19), we have the assertion (21).

Estimates of the weak and strong type

Theorem 2. Under the conditions of Theorem 1 the estimates of weak type

$$\mu\{x\in Q \mid (Tf)(x) > \varsigma > 0\} \le C_{p,\lambda}\left(\frac{\|f\|_p}{\varsigma}\right)^p,\quad p\ge 1$$

and strong type

$$\|Tf\|_p \le C_{p,\lambda}\|f\|_p,\quad p>1;$$

$$\|Tf\| \le C_\lambda\left(1+\left\|f\left(\ln^+|f|\right)\right\|\right);$$

$$\|Tf\|_p \le C_{p,\lambda}\|f\|,\quad 0<p<1$$

hold if Tf are any of the operators (15) – (17).

The assertion of theorem for the operators (16), (17) follows from Theorem 1 and the corresponding estimates for (12), (13) cited in paragraph 2. The assertion for the operator (15) now follows from (11).

Non-tangential summability

Theorem 3. If the sequence (5) satisfies the conditions (18), (19) and

$$\lim_{h\to 0}\lambda_k(h)=1,\ \ k=0,1,\dots\ , \tag{35}$$

then the relations

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} U(f,y;\lambda,h)=f(x)\,, \tag{36}$$

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} \widetilde{U}(f,y;\lambda,h)=\widetilde{f}(x) \tag{37}$$

hold almost everywhere for each $f\in L(Q)$. Under the same conditions on the sequence (5) the equality

$$\lim_{\substack{(y,h)\to(x,0)\\(y,h)\in\Gamma_d(x)}} \theta(\varphi,y;\lambda,h)=\varphi(\exp(ix)) \tag{38}$$

is valid for each $\varphi\in H$ and almost all x.

The relations (36), (37) follow from the weak type estimates (Theorem 2) and condition (35) by the standard method [1, vol. 2, p. 464-465]. The relation (38) follows from (11), (36) and (37), when you consider that the limit of the left side (38) almost everywhere is

$$f(x)+i\,\widetilde{f}(x)\,,$$

which is equal $\varphi(\exp(ix))$ by (2) и (11).

Piecewise convex summation methods

It was noted in [3] that under the conditions (18) every piecewise-convex sequence (5) satisfies the condition

$$\sum_{k=1}^{\infty} k\left|\Delta^2\lambda_k(h)\right|\le C_\lambda\,.$$

By virtue of piecewise convex sequence (5) the second finite differences $\Delta^2\lambda_k(h)$ retain the sign. Suppose for definiteness, it will be a plus sign at all sufficiently large k (depending, generally speaking, from h), namely $k\ge\tau(m)$, where $\tau=\tau(m)$ – some positive integer,

$$\tau=\tau(m)=\tau(m,\lambda)\le m\,. \tag{39}$$

The sum (19) does not exceed

$$C_\lambda\left(\sum_{k=1}^{\infty} k\left|\Delta^2\lambda_k(h)\right|+\sum_{k=m}^{\infty}\frac{k^2}{m}\left|\Delta^2\lambda_k(h)\right|\right). \tag{40}$$

In the second sum in (40) all $\Delta^2\lambda_k(h)$ are positive by (39); applying twice Abel transform, we have

$$\sum_{k=m}^{\infty}\left|\Delta^2\lambda_k(h)\right|\frac{k^2}{m}=\frac{1}{m}\sum_{k=m}^{\infty}k^2\Delta^2\lambda_k(h)=\frac{1}{m}\left(m^2\Delta^2\lambda_m(h)+\sum_{k=m+1}^{\infty}(2k-1)\Delta\,\lambda_k(h)\right)=$$

$$=m\Delta^2\lambda_m(h)+\frac{2m+1}{m}\Delta\lambda_{m+1}(h)+\frac{1}{m}\sum_{k=m+2}^{\infty}\lambda_k(h)\ .$$

Thus, under conditions (18) and

$$\frac{1}{m}\sum_{k=m+2}^{\infty}\left|\lambda_k(h)\right|\ \le C_\lambda\,, \tag{41}$$

the assertions of Theorems 2 and 3 are valid for each piecewise-convex sequence (5).

Exponential summation methods

Summation methods

$$\lambda_0(h)=1\,,\ \lambda_k(h)=\lambda(x,h)\,|_{x=k}\ ,\quad k=1,2,...,\ \text{where}\ \lambda(x,h)=\exp(-h\varphi(x))\,,$$

were studied in [3] in the case of "radial" convergence; in particular, it was given the condition of piecewise convexity of sequence $\{\lambda_k(h)\}$. In this paper we consider

$$\lambda(x,h)=\exp\left(-hx^\alpha\right),\ \alpha\ge 1. \tag{42}$$

This function is a piecewise-convex, since

$$\lambda'(x,h)=-\alpha h\,x^{\alpha-1}\,,\quad \lambda''(x,h)=\alpha h\,x^{\alpha-2}\left(h\alpha\,x^\alpha-(\alpha-1)\right) \tag{43}$$

and the second derivative (43) changes its sign once. Consequently, the sequence

$$\lambda_k(h)=\ \exp\left(-hk^\alpha\right),\ \ \alpha\ge 1\,, \tag{44}$$

is a piecewise-convex too, and the second finite differences have constant sign for all

$$k\ge\left(\frac{2d(\alpha-1)}{\alpha}\right)^{\frac{1}{\alpha}}m^{\frac{1}{\alpha}}\,,\ \text{so that}\ \ \tau=1+\left[\left(\frac{2d(\alpha-1)}{\alpha}\right)^{\frac{1}{\alpha}}m^{\frac{1}{\alpha}}\right];$$

at the same time $\tau\le m$ for sufficiently large m.

It is obvious that condition (35) is valid for the sequence (44), and is also easy to verify the conditions (18), since (by Lagrange's theorem)

$$\Delta\lambda_k(h)=\alpha h(k+\theta)^{\alpha-1}\exp(-h(k+\theta)^\alpha)\,,\ \text{where}\ \ \theta=\theta(k)\in(0,1)\,.$$

We verify the satisfiability of condition (41). Because of decrease of the function (42) we have a sum in (41) not exceeding

$$\frac{1}{m}\sum_{k=2}^{\infty}|\,\lambda_k(h)\,|\ \le\frac{1}{m}\int_0^\infty\exp\left(-hx^\alpha\right)dx=\frac{1}{\alpha\,m}h^{-\frac{1}{\alpha}}\int_0^\infty t^{\frac{1}{\alpha}-1}\ \exp(-t)dt\le C_\alpha h^{1-\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}\right), \tag{45}$$

where $\Gamma = \Gamma\left(\frac{1}{\alpha}\right)$ – Euler gamma function. For $\alpha \geq 1$ the right side of (45) does not exceed a constant that depends only on α. Thus, Theorems 2 and 3 are valid for exponential summation methods (44).

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Некасательная суммируемость степенных разложений функций классов Харди

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Ключевые слова: максимальные операторы; некасательная суммируемость; оценки слабого и сильного типов.

Аннотация: Построен класс H_{λ}^{p} обобщенных λ-средних степенных рядов аналитических функций $\varphi \in H^{p}$. Изучено поведение λ-средних при стремлении произвольной точки круга (y, r) к точке $(x, 1)$ по некасательным направлениям. Получены оценки соответствующих максимальных операторов и теоремы о суммируемости почти всюду. В основе результатов лежит представление $H_{\lambda}^{p} = L_{\lambda}^{p} \oplus i\,\widetilde{L}_{\lambda}^{p}$, в котором L_{λ}^{p} и $\widetilde{L}_{\lambda}^{p}$ соответственно, классы λ-средних рядов Фурье и сопряженных рядов функций $f = \mathrm{Re}\,\varphi$.

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Nichttangentiale Summierung der Potenzzerlegungen der Funktionen der Hardy-Klasse

Zusammenfassung: Es ist die Klasse H_{λ}^{p} der verallgemeinerten λ-mittleren Potenzreihen der analytischen Funktionen $\varphi \in H^{p}$ aufgebaut. Es ist das Verhalten von λ-mittleren beim Streben eines willkürlichen Punktes des Kreises (y, r) zum Punkt $(x, 1)$ nach den nicht Tangentialrichtungen erlernt. Es sind die Einschätzungen der entsprechenden maximalen Operatoren und des Theorems über der Summierung fast überall erhalten. Zugrunde der Ergebnisse liegt die Beibringung $H_{\lambda}^{p} = L_{\lambda}^{p} \oplus i\,\tilde{L}_{\lambda}^{p}$, in der L_{λ}^{p} и $\tilde{L}_{\lambda}^{p}$, entsprechend, die Klassen der λ-mittleren Reihen von Foirier und der verknüpften Reihen der Funktionen $f = \mathrm{Re}\varphi$.

Sommation non-tangentiel des extensions de puissance des fonctions de classe Hardy

Résumé: Est construite la classe H_{λ}^{p} des séries moyennes de puissance des fonctions analytiques . Est étudié le comportement des moyennes λ lors de la quête de l'arbitraire d'un point d'un cercle (y, r) à un point $(x, 1)$ par les directions non-tangentielles. Sont obtenues les évaluations des opérateurs maximums correspondants et du théorème de la sommation presque partout. A la base des résultats se trouve la représentation $H_{\lambda}^{p} = L_{\lambda}^{p} \oplus i\,\tilde{L}_{\lambda}^{p}$, dans laquelle L_{λ}^{p} и $\tilde{L}_{\lambda}^{p}$ sont respectivement les classes des séries moyennes de Fourier et des séries de fonctions $f = \mathrm{Re}\varphi$.

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