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WEIGHTED NORM INEQUALITIES FOR THE CONVOLUTION OPERATORS

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Abstract: Weighted norm inequalities for Steklov integral means $f_h(x)$ and for the set of convolution operators are obtained. The base of results is a relation

$$|(f*K)(x)| \le C \int\limits_{[0,\pi]^N} K^*(h) f_{4h}(x)dh,$$

where an even kernel $K(x)$ possesses on $[0,\pi]^N$ nonincreasing on each variable majorant $K^*(x)$.

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1. Introduction

Let $x, t, h, ...$ be points of real N-dimensional space and functions $f = f(x)$, $u = u(x) \ge 0$, $v = v(x) \ge 0$ are measurable on the square $Q_N = (-\pi, \pi]^N$ and 2π-periodic on every variable x_j, $j = 1, 2, ..., N$. Define

$$A_p(u,v;\Omega) = \left(\frac{1}{|\Omega|}\int_\Omega u(t)dt\right)\left(\frac{1}{|\Omega|}\int_\Omega v^{-1/(p-1)}(t)dt\right)^{p-1}, \quad p>1$$

and

$$A_1(u,v;\Omega) = \left(\frac{1}{|\Omega|}\int_\Omega u(t)dt\right)\operatorname{esssup}_{t\in\Omega}\frac{1}{v(t)}$$

for some rectangle $\Omega = \otimes_{j=1}^N [a_j, b_j]$. In the present paper we study a problem of the boundedness of Steklov integral means

$$f_h(x) = |\Omega(h)|^{-1}\int_{\Omega(h)} |f(x+t)|\,dt$$

(over rectangles $\Omega(h) = \otimes_{j=1}^N [-h_j, h_j]$) and the set of convolution operators

$$(f*K_\varepsilon)(x) = \int_{Q^N} f(t) K_\varepsilon(x-t)dt,$$

from $L_v^p(Q^N)$ in $L_u^p(Q^N)$.

Our results prolong the researches of [1], where B. Muckenhoupt obtained (for $N=1$) the necessity and sufficiency of the condition

$$\sup_{\Omega} A_p(u,v;\Omega)<\infty \tag{1.1}$$

(so-called A_p-condition) for the boundedness of Poisson integral as operator from $L_v^p(Q^1)$ in $L_u^p(Q^1)$, $p\ge 1$.

Let $0\cdot\infty$ will equal 0 and C will represent a constant, though not necessarily one such constant.

2. Steklov integral means

The following statement extends one of the results [2].

Theorem 1. If $p\ge 1$, then the statements (1.1) and

$$\sup_h \int_{Q^N}(f_h(x))^p u(x)dx \le C\int_{Q^N}|f(x)|^p\, v(x)dx \tag{2.1}$$

(where C depends on $u,\ v,\ p,\ N$) are equivalent.

P r o o f. For the sufficiency of A_p condition (1.1) we proof firstly the inequality

$$\sup_h \int_{Q^N}(f_h(x))^p u(x)dx \le C\left(\sup_{\Omega} A_p(u,v;\Omega)\right)\int_{Q^N}|f(x)|^p\, v(x)dx,\ \ p\ge 1, \tag{2.2}$$

where C depends on p and N only; the case of $N=1$ was proved by B. Muckenhoupt (see [1]). Assume $f\in L_v^p(Q^N)$ (i.e. the right part of (2.2) is finite), otherwise (2.2) is trivial.

Except the trivial cases of $u(x)=0$ almost everywhere in Q^N and

$$\int_{\Omega} v^{-1/(p-1)}(x)dx=\infty$$

(for some Ω), since in this case $v(x)=0$ for almost every $x\in\Omega$, and then (see (1.1)) the relation $u(x)=0$ holds almost everywhere in Q^N.

It is possible to be restricted of a case $0<h_j\le\pi,\ j=1,...,N$ only; otherwise (let, for example, $j=1$) there exists natural number l, such as $l\pi\le h_1<(l+1)\pi$ and

$$\frac{1}{h_1}\int_{-h_1}^{h_1}...\int_{-h_N}^{h_N}|f(x+t)|\,dt\le\frac{1}{l\pi}\int_{-(l+1)\pi}^{(l+1)\pi}...\int_{-h_N}^{h_N}|f(x+t)|\,dt=\frac{l+1}{l}f_{\pi,...,h_N}(x).$$

Let's assume

$$\Omega_k(h)=\otimes_{j=1}^N[k_jh_j,(k_j+1)h_j],\quad \Omega_k'(h)=\otimes_{j=1}^N[(k_j-1)h_j,(k_j+2)h_j]$$

and concider natural numbers P_j, such that $\pi/h_j\le P_j<\pi/h_j+1,\ j=1,...,N.$ If $p>1$, then by Holder's inequality we have for each h

$$\int_{Q^N}(f_h(x))^p u(x)dx\le$$

$$\le\sum_{|k_1|\le P_1}...\sum_{|k_N|\le P_N}\int_{\Omega_k(h)}\left(\frac{1}{|\Omega(h)|}\int_{x+\Omega(h)}|f(t)|\,dt\right)^p u(x)dx\le$$

$$\le \sum_{|k_1|\le P_1} \dots \sum_{|k_N|\le P_N} \left(\frac{1}{|\Omega(h)|}\right)^p \left(\int_{\Omega_k(h)} u(x)dx\right)\left(\int_{\Omega_{k'}(h)} |f(t)|\, v^{1/p} v^{-1/p} dt\right)^p \le$$

$$\le C \sum_{|k_1|\le P_1} \dots \sum_{|k_N|\le P_N} \left(\frac{1}{|\Omega'_k(h)|}\int_{\Omega'_k(h)} u(x)dx\right)\left(\frac{1}{|\Omega'_k(h)|}\int_{\Omega'_k(h)} v^{-1/(p-1)}(x)dx\right)^{p-1} \times$$

$$\times \int_{\Omega'_k(h)} |f(t)|^p\, v(t)dt \le$$

$$\le C\left(\sup_{\Omega} A_p(u,v;\Omega)\right) \sum_{|k_1|\le P_1} \dots \sum_{|k_N|\le P_N} \int_{\Omega_{k'}(h)} |f(t)|^p\, v(t)dt. \qquad (2.3)$$

Since all intervals $[(k_j-1)h_j, (k_j+2)h_j]$ are subset of $[-4\pi, 4\pi]$, and no point is in more than three of these intervals, the inequality (2.2) follows from (2.3) for $p>1$. Replacing

$$\left(\int_{\Omega_k(h)} v^{-1/(p-1)}(x)dx\right)^{p-1}$$

by

$$\mathrm{esssup}_{t\in\Omega} \frac{1}{v(t)}$$

in the above-stated reasonings, we obtain (2.2) in the case $p=1$.

For the necessity of the A_p-condition we use arguments of [1]. Denote

$$V = \int_{\Omega} v^{-1/(p-1)}(x)dx$$

for $p>1$ and choose some rectangle $\Omega = \otimes_{j=1}^{N}[a_j, b_j]$, where the length of each side $b_j - a_j \le 2\pi$; the case $b_j - a_j > 2\pi$ for some j one can deduce exactly in the same way, as above. So, there exists some vector b^0, such that $Q_0^N = b^0 + Q^N$ and $\Omega \subset Q_0^N$. If $V=0$, then (1.1) is true. In the cases of $V=\infty$ and $0<V<\infty$ denote $\eta_j = (b_j - a_j)/2,\ j=1,\dots,N$ and represent Ω in the form $\Omega = \otimes_{j=1}^{N}[\alpha_j - \eta_j, \alpha_j + \eta_j]$, where α_j are the centres of appropriate intervals. Then $\Omega \subset x + \Omega(2\eta)$ for all $x \in \Omega$ and

$$f_{2\eta}(x) = |\Omega(2\eta)|^{-1} \int_{x+\Omega(2\eta)} |f(t)|\,dt \ge 2^{-N} |\Omega|^{-1} \int_{\Omega} |f(t)|\,dt.$$

If $V=\infty, v^{-1/p}$ is not in $L^{p'}(\Omega)$, where $p' = p/(p-1)$. So there is a function $g \in L^p(\Omega)$, and $g=0$ on $Q_0^N \setminus \Omega$, such that

$$\int_{\Omega} g v^{-1/p} dx = \infty. \qquad (2.4)$$

Let $f = gv^{-1/p}$. Since $f^p v = g^p, f^p v$ is integrable on Ω while for $x\in\Omega$ the relation $f_{2\eta}(x) = \infty$ holds by (2.4). Therefore, (2.1) implies that $u=0$ almost everywhere in and (1.1) is true.

If $0 < V < \infty$, let $f = v^{-1/(p-1)}\chi_\Omega$; here $\chi_\Omega = \chi_\Omega(x)$ is a characteristic set function. It follows from (2.4) that

$$\sup_h \int_{Q^N} (f_h(x))^p u(x)dx \geq \left(2^{-N}\frac{V}{|\Omega|}\right)^p \int_\Omega udx,$$

and we have from (2.1)

$$\left(\frac{V}{|\Omega|}\right)^p \int_\Omega udx \leq C\int_\Omega v^{-p/(p-1)}vdx. \tag{2.5}$$

Multiplying both sides of (2.5) by V^{-1} proves (1.1) for $p > 1$.

Let $p = 1$ now. For $\operatorname{essinf}_{x\in\Omega} v(x) = \infty$ the relation (1.1) is obvious; otherwise for every $\varepsilon > 0$ there exists a measurable $E \subset \Omega$, $|E| > 0$ such that $v(x) < \varepsilon + \operatorname{essinf}_{x\in\Omega} v(x)$ for all $x \in E$. Define $f = 1$ on Ω and $f = 0$, $x \in Q_0^N \setminus \Omega$. Then

$$\sup_h \int_{Q^N} f_h(x)u(x)dx \geq \frac{1}{|\Omega(2\eta)|}\int_{x+\Omega(2\eta)} |f(t)|\,dt \geq 2^{-N}\frac{1}{|\Omega|}|E|$$

for all $x \in \Omega$ and we have from (2.1)

$$\frac{1}{|\Omega|}|E|\int_\Omega udx \leq C|E|(\varepsilon + \operatorname{essinf}_{x\in\Omega} v(x)). \tag{2.6}$$

Since ε is an arbitrary, the A_1-condition follows from (2.6).

3. Estimates for the convolution operators

Let now $K(x)$ be a measurable even function, possessing even nonincreasing till each variable $x_j \in [0,\pi]$ and 2π-periodic (on each variable) majorant $K^* = K^*(x)$, i.e. $|K(x)| \leq |K^*(x)|$, $x \in Q^N$.

Theorem 2. The relation

$$|(f * K)(x)| \leq C\int_{[0,\pi]^N} K^*(h) f_{4h}(x)dh$$

holds for every x; the constant C depends on N only.

P r o o f. By the evenness of $K^*(x)$ it is sufficient to obtain the upper estimate for

$$J_\delta(x) = \int_{[\delta,\pi]^N} |f(x+t)|\,K^*(t)dt,$$

where $0 < \delta \leq \pi$. Fix some j and consider

$$\int_\delta^\pi |f(\ldots, x_j + t_j, \ldots)|\,K^*(\ldots, t_j, \ldots)dt_j. \tag{3.1}$$

Denote $\chi = x_j, \tau = t_j$, and define natural number T such that

$$\log_2 \pi/\delta \leq T < 1 + \log_2 \pi/\delta.$$

Since $K^*(...,\tau,...)$ is nonincreasing on a variable τ we obtain

$$\int_{\delta}^{\pi}|f(...,\chi+\tau,...)|K^*(...,\tau,...)d\tau\le\sum_{\nu=1}^{T-1}\int_{2^{\nu-1}\delta}^{2^{\nu}\delta}|f(...,\chi+\tau,...)|K^*(...,\tau,...)d\tau+$$

$$+\int_{2^{T-1}\delta}^{\pi}|f(...,\chi+\tau,...)|K^*(...,\tau,...)\,d\tau\le$$

$$\le\sum_{\nu=1}^{T}K^*(...,2^{\nu-1}\delta,...)\int_0^{2^{\nu}\delta}|f(...,\chi+\tau,...)|\,d\tau. \qquad (3.2)$$

Denote

$$\Phi_{\chi}(r)=\int_0^r|f(...,\chi+\tau,...)|\,d\tau. \qquad (3.3)$$

Then the integral (3.1) is no more (see (3.2)) than

$$2\sum_{\nu=1}^{T}\frac{K^*\left(...,\frac{1}{2}2^{\nu}\delta,...\right)}{2^{\nu}\delta}\Phi_{\chi}(2^{\nu}\delta)(2^{\nu}\delta-2^{\nu-1}\delta).$$

Now

$$\frac{K^*\left(...,\frac{1}{2}r,...\right)}{r}\Phi_{\chi}(2r),\ \ r\in[\delta,\pi]$$

is continuous function and we can obtain the following relation from a theorem of the mean value :

$$\int_{2^{\nu-1}\delta}^{2^{\nu}\delta}\frac{K^*\left(...,\frac{1}{2}r,...\right)}{r}\Phi_{\chi}(2r)dr=\frac{K^*\left(...,\frac{1}{2}r_{\nu},...\right)}{r_{\nu}}\Phi_{\chi}(2r_{\nu})(2^{\nu}\delta-2^{\nu-1}\delta), \quad (3.4)$$

where $r_{\nu}\in(2^{\nu-1}\delta,\,2^{\nu}\delta)$. Since $2r_{\nu}\ge2^{\nu}\delta$, we have

$$\frac{K^*\left(...,\frac{1}{2}r_{\nu},...\right)}{r_{\nu}}\Phi_{\chi}(2r_{\nu})(2^{\nu}\delta-2^{\nu-1}\delta)\le$$

$$\ge\frac{K^*\left(...,\frac{1}{2}2^{\nu}\delta,...\right)}{2^{\nu}\delta}\Phi_{\chi}(2^{\nu}\delta)(2^{\nu}\delta-2^{\nu-1}\delta). \qquad (3.5)$$

According to relations (3.2), (3.3), (3.5), (3.4) the left part of (3.2) is no more, then

$$C\sum_{\nu=1}^{T}\int_{2^{\nu-1}\delta}^{2^{\nu}\delta}\frac{K^*\left(...,\frac{1}{2}r,...\right)}{r}\Phi_{\chi}(2r)dr\le$$

$$\le C\int_0^{2\pi}K^*\left(...,\frac{1}{2}r,...\right)\left(\frac{1}{2r}\int_0^{2r}|f(...,\chi+\tau,...)|\,d\tau\right)dr.$$

Fulfilling sequentially the indicated estimations for integrals of the type (3.1) on each variable, we obtain

$$J_{\delta}(x)\le C\int_{[0,2\pi]^N}K^*\left(\frac{1}{2}h\right)f_{2h}(x)dh=C\int_{[0,\pi]^N}K^*(h)f_{4h}(x)dh. \qquad (3.6)$$

The assertion of theorem follows from (3.6) because δ is arbitrary.

Let the kernels $K_m(x)$, $m \in Z_+^N$, and their majorants $K_m^* = K_m^*(x)$ satisfy the above-stated conditions.

The inequality

$$\int_{Q^N} |(f * K_m)(x)|^p \, u(x)dx \le$$

$$\le C\left(\int_{[0,\pi]^N} K_m^*(t)dt\right)\left(\sup_h A_p(u,v;h)\right)\int_{Q^N} |f(x)|^p \, v(x)dx \qquad (3.7)$$

holds for every $p \ge 1$; the constant C depends on p and N only. This follows immediately from Theorems 1 (see (2.1)) and 2 and Minkowski integral inequality.

Note, that a sequence of linear means of multiple Fourier sums possesses the integral form just of $f * K_m = (f * K_m)(x)$.

4. Weighted norm inequalities for some linear means of multiple Fourier series

Let now $\{K_\varepsilon\}$ be a set of integral kernels, possessing the properties of p. 3 and for each Ω there exists $\varepsilon = \{\varepsilon_1, ..., \varepsilon_N\}, \varepsilon_j > 0, j = 1, ..., N$ such as

$$K_\varepsilon(x - t) \ge C/|\Omega|; \qquad x, t \in \Omega. \qquad (4.1)$$

Let

$$\sup_\varepsilon \int_{[0,\pi]^N} K_\varepsilon^*(x)dx < \infty \qquad (4.2)$$

for the the sequence of corresponding majorants $K_\varepsilon^*(x)$. Define

$$\sigma_\varepsilon(f, x) = (f^* K_\varepsilon)(x).$$

Theorem 3. If the relations (4.1), (4.2) hold, then the statements (1.1) and

$$\sup_\varepsilon \int_{Q^N} |\sigma_\varepsilon(f,x)|^p \, u(x)dx \le C\int_{Q^N} |f(x)|^p \, v(x)dx$$

(where C depends on u, v, p, N) are equivalent for every $p \ge 1$.

P r o o f. For the sufficiency of A_p condition (1.1) we use the results (3.7) and (4.2). The necessity of the A_p-condition may be deduce by using (4.1) and choosing f such as in the prove of Theorem 1.

The set of integral kernels

$$K_\varepsilon(t) = K_{\varepsilon,\alpha}(t) = \pi^{-N} \prod_{j=1}^{N} \mathrm{K}_{\varepsilon_j,\alpha_j}(t_j), \quad 0 < \alpha_j \le 1, \ j = 1, ..., N,$$

where

$$\mathrm{K}_{\varsigma,\beta}(\tau) = \frac{1}{2} + \sum_{k=0}^{\infty} \exp(-\varsigma k^\beta)\cos k\tau,$$

generates the convolution operators of the generalized Poisson type. This set satisfies to conditons (4.2) and (4.1). The validity of (4.2) follows from the representation

$$K_{\varsigma,\beta}(\tau)=\sum_{k=0}^{\infty}\Delta^2\exp(-\varsigma k^{\beta})\frac{\sin^2(k+1)\frac{\tau}{2}}{2\sin^2\frac{\tau}{2}}$$

and the property of convexity of sequence $\lambda_k^{\varsigma,\beta} = \exp(-\varsigma k^{\beta})$.

For each $\tau\in I,\ |I|\le 2\pi$ the relation (4.1) follows from the inequality

$$K_{\varsigma,\beta}(\tau)\ge Cm^2\sum_{k=m}^{2m-1}\Delta^2\exp(-\varsigma k^{\beta})=$$

$$=Cm^2(\Delta\exp(-\varsigma m^{\beta})-\Delta\exp(-\varsigma(2m)^{\beta}))\ge Cm(\varsigma m^{\beta}),$$

if we choose $m=\left[\frac{\pi}{2|I|}\right]$ and $\varsigma=m^{-\beta}$, $0<\beta\le 1$; the constant $C>0$ depends on β only.

Hence the assertion of Theorem 3 remains valid for the set of convolutions $\sigma_{\varepsilon,\alpha}(f,x)=(f*K_{\varepsilon,\alpha})(x)$.

Note, that the such type statement remains valid for some class of the linear means of multiple Fourier sums over oblique parallelepipeds.

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Оценки весовых норм операторов свертки

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Аннотация: Получены оценки весовых норм интегральных средних Стеклова $f_h(x)$ и операторов свертки. Результаты основаны на оценке

$$|(f*K)(x)|\le C\int_{[0,\pi]^N}K^*(h)f_{4h}(x)dh,$$

где четное ядро $K(x)$ обладает на $[0,\pi]^N$ невозрастающей по каждой переменной мажорантой $K^*(x)$.

Einschätzungen der Gewichtsnormen der Faltungsoperatoren

Zusammenfassung: Es sind die Einschätzungen der Gewichtsnormen der Integralmittel von Steklow f_h(x) und der Faltungsoperatoren erhalten. Die Resultate gründen sich auf der Einschätzung

$$|(f*K)(x)| \le C \int_{[0,\pi]^N} K^*(h) f_{4h}(x) dh,$$

wo gerader Kern $K(x)$ besitzt auf $[0,\pi]^N$ unzunehmendem auf jedem Wechselmajorant $K^*(x)$.

Appréciations des normes de poids des opérateurs de la convolution

Résumé: Sont obtenues les appréciations des normes de poids des moyennes de Steklov $f_h(x)$ et des opérateurs de la convolution. Les résultats sont fondés sur la relation

$$|(f*K)(x)| \le C \int_{[0,\pi]^N} K^*(h) f_{4h}(x) dh,$$

Où le noyau pair $K(x)$ possède sur $[0,\pi]^N$ de la croissante par chaque majorant variable $K^*(x)$.

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